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On the rank of a symmetric form ☆

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ABSTRACT

We give a lower bound for the degree of a finite apolar subscheme of a complex symmetric form F, in terms of the degrees of the generators of the annihilator ideal $F^{\perp}$. In the special case, when F is a monomial $x_0^{d_0} \cdot x_1^{d_1} \cdot \ldots \cdot x_n^{d_n}$ with $d_0 \leqslant d_1 \leqslant \cdots \leqslant d_{n-1} \leqslant d_n$ we deduce that the minimal length of an apolar subscheme of F is $(d_0+1) \cdot \ldots \cdot (d_{n-1}+1)$, and if $d_0 = \cdots = d_n$, then this minimal length coincides with the rank of F.

Let $F \in T = \mathbb{C}[x_0, \ldots, x_n]$ be a homogeneous form and let $S = \mathbb{C}[y_0, \ldots, y_n]$ be the ring of commuting differential operators acting on T. The action is called apolarity, and we identify S and T with the homogeneous coordinate rings of $\mathbb{P}^n$ and of the dual space $\check{\mathbb{P}}^n$ respectively, as in [6, 1.3]. The annihilator of F is an ideal $F^{\perp} \subset S$. A finite subscheme $\Gamma \subset \mathbb{P}^n$ is apolar to F if the homogeneous ideal $I_\Gamma \subset S$ is contained in $F^{\perp}$.

We define the cactus rank $cr(F)$ as

$$cr(F) = \min\{\deg \Gamma \mid \Gamma \subset \mathbb{P}^n,\ \dim \Gamma = 0,\ I_\Gamma \subset F^{\perp}\},$$

the smoothable rank $sr(F)$ as

$$sr(F) = \min\{\deg \Gamma \mid \Gamma \subset \mathbb{P}^n \text{ smoothable},\ \dim \Gamma = 0,\ I_\Gamma \subset F^{\perp}\}$$

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and the rank $r(F)$ as

$$r(F) = \min\{\deg \Gamma \mid \Gamma \subset \mathbb{P}^n \text{ smooth, } \dim \Gamma = 0,\ I_\Gamma \subset F^\perp\}.$$

Here Γ is smoothable if it can be deformed to a smooth scheme as in [3, III, 9]. Clearly $cr(F) \leqslant sr(F) \leqslant r(F)$. We shall give lower bounds for these ranks in terms of the generators of the ideal $F^\perp$. The related notion of border rank, $br(F)$, is defined as the minimal k such that $[F]$ lies in the Zariski closure of the set of forms of rank k in the projective space $\mathbb{P}(T_\delta)$ of forms of degree $\delta = \deg F$. In general $br(F) \leqslant sr(F)$, so our lower bounds for $sr(F)$ does not apply unconditionally to $br(F)$. Notice also that cactus rank coincides with the notion of scheme length as defined by Iarrobino [5, Definition 4D]. For applications of these notions of rank to powersum decompositions of symmetric forms and to equations of secant varieties see [6], [4] and [1], the latter inspired our use of the name cactus rank.

We define the degree of $F^\perp$ to be the length of the quotient algebra $S_F = S/F^\perp$.

Proposition 1. *If the ideal of $F^\perp$ is generated in degree d and $\Gamma \subset \mathbb{P}^n$ is a finite apolar subscheme to F, then*

$$\deg \Gamma \geqslant \frac{1}{d} \deg F^\perp.$$

Proof. Consider the affine cones in $\mathbb{A}^{n+1}$ defined by the homogeneous ideals $F^\perp$ and I_Γ. Taking the projective closures in $\mathbb{P}^{n+1}$, we may assume that $F^\perp$ and I_Γ define subschemes X and Y of pure dimension 0 and 1 in $\mathbb{P}^{n+1}$. Furthermore $\deg Y = \deg \Gamma$ and $\deg X = \deg F^\perp$. The apolarity condition says that $I_X \supset I_Y$, i.e. that $X \subset Y$ as schemes. Now, take an element g in I_X that does not contain any component of Y. Then the hypersurface $G = \{g = 0\}$ has proper intersection with Y and contains X. Therefore, by Bézout's theorem [2, 8.8.4],

$$\deg G \cdot \deg Y \geqslant \deg X.$$

The proposition follows by taking g of degree d. $\square$

Corollary 1. *If the ideal $F^\perp$ is generated in degree d, then the cactus rank*

$$cr(F) \geqslant \frac{1}{d} \deg F^\perp.$$

Corollary 2. *If F is a monomial, $F = x_0^{d_0} \cdot x_1^{d_1} \cdot \ldots \cdot x_n^{d_n}$ with $d_0 \leqslant d_1 \leqslant \cdots \leqslant d_n$, then the cactus rank and the smoothable rank coincide and equals*

$$cr(F) = sr(F) = (d_0 + 1) \cdot \ldots \cdot (d_{n-1} + 1).$$

If furthermore $d_n = d_0 = d$, i.e. $F = (x_0 \cdot x_1 \cdot \ldots \cdot x_n)^d$, then $r(F) = cr(F) = sr(F) = (d + 1)^n$.

Proof. When $F = x_0^{d_0} \cdot x_1^{d_1} \cdot \ldots \cdot x_n^{d_n}$, then $F^\perp$ is the complete intersection generated by the forms

$$y_0^{d_0+1}, y_1^{d_1+1}, \ldots, y_n^{d_n+1}.$$

So it is generated in degree $d_n + 1$, while $F^\perp$ has degree

$$(d_0 + 1) \cdot (d_1 + 1) \cdot \ldots \cdot (d_n + 1).$$

The formula for the cactus rank follows, since the first n generators define a finite apolar subscheme of degree $(d_0+1)\cdot\ldots\cdot(d_{n-1}+1)$. Now, any complete intersection is smoothable, by deforming each generator separately and using Bertini's theorem [3, II, 8.18], so the smoothable rank equals the cactus rank for F. If $d_0 = d_n = d$, then the forms of degree $d+1$ in $F^\perp$ have no basepoints so, by Bertini's theorem, n general forms in $F^\perp$ of degree $(d+1)$ define a smooth finite subscheme of degree $(d+1)^n$ in $\mathbb{P}^n$. $\square$

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